



CareerUdaya
— Rise with Purpose —

Class 12 Mathematics

Part I

Complete Study Notes

CBSE

AHSEC

JEE

Engineering Entrance

Competitive Exams

Rise with Purpose

PREPARED BY

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CHAPTER 1

Relations and Functions

Introduction

A relation is simply a rule that links the members of one set to the members of another. A function is a special, well-behaved relation in which every input is sent to exactly one output. In Class 12 we go deeper: we classify relations by the properties they satisfy and we study how functions can be combined and reversed. These ideas are the grammar of higher mathematics — limits, calculus, linear algebra and probability are all built on top of them.

Why This Chapter Matters

Almost every advanced topic you will meet — derivatives, integrals, matrices, vectors, probability distributions — is phrased in the language of functions. Understanding one-one and onto behaviour tells you when a process can be undone (inverse functions), which is the foundation of logarithms, inverse trigonometry and cryptography.

Real-Life Applications

- A password system maps each user to one unique login token — a one-one (injective) function; no two users may share a token.
- An equivalence relation groups objects that behave the same way: blood-group classification, modular arithmetic in computing (clock time, hashing), and grouping triangles by similarity.
- GPS coordinates assign each location exactly one (latitude, longitude) pair — a function from places to number pairs.

Key Concepts

Relation

For two sets A and B , any subset R of the Cartesian product $A \times B$ is a relation from A to B . If the ordered pair (a, b) lies in R we write $a R b$ and say 'a is related to b'. A relation on a single set A is a subset of $A \times A$.

Empty and Universal relations

If no element is related to any other, $R = \phi$ (empty relation). If every element is related to every other, $R = A \times A$ (universal relation). Both are called trivial relations.

Reflexive, Symmetric, Transitive

Reflexive: (a, a) is in R for every a . Symmetric: whenever (a, b) is in R , (b, a) is also in R . Transitive: whenever (a, b) and (b, c) are in R , (a, c) is in R as well.

Equivalence relation and classes

A relation that is reflexive, symmetric AND transitive is an equivalence relation. It splits the set into disjoint equivalence classes $[a]$ — buckets in which every member is related to every other member, and members of different buckets are never related.

Function

A function $f : A \rightarrow B$ assigns to each element of A exactly one element of B . A is the domain, B the co-domain, and the set of actual outputs is the range (range is a subset of the co-domain).

One-one, Onto, Bijective

One-one (injective): different inputs give different outputs, i.e. $f(x_1) = f(x_2)$ forces $x_1 = x_2$. Onto (surjective): every element of B is hit, i.e. range = co-domain. Bijective: both one-one and onto.

Composition of functions

If $f : A \rightarrow B$ and $g : B \rightarrow C$, the composite $g \circ f : A \rightarrow C$ is defined by $(g \circ f)(x) = g(f(x))$. Order matters: in general $f \circ g$ is not equal to $g \circ f$.

Invertible function

$f : X \rightarrow Y$ is invertible if there is $g : Y \rightarrow X$ with $g \circ f = \text{identity on } X$ and $f \circ g = \text{identity on } Y$. A function is invertible exactly when it is bijective; its inverse is written f^{-1} .

Definitions

Cartesian product $A \times B$	Set of all ordered pairs (a, b) with a in A and b in B .
Equivalence class $[a]$	All elements related to a under an equivalence relation R .
Identity function I_X	$I_X(x) = x$ for every x ; it leaves every input unchanged.
Inverse function f^{-1}	The unique function that reverses f when f is bijective.

Formula / Key Results Table

Number of relations on a set of n elements	2^{n^2}
Number of functions from set of m to set of n elements	n^m
Number of one-one functions ($m \leq n$)	$n! / (n - m)!$
Number of bijections from a set of n to itself	$n!$
Composition	$(g \circ f)(x) = g(f(x))$
Inverse condition	$g \circ f = I_X$ and $f \circ g = I_Y \Rightarrow g = f^{-1}$

Important Properties

- On a FINITE set X , a function $f : X \rightarrow X$ is one-one if and only if it is onto. This fails for infinite sets (e.g. $f(x) = 2x$ on \mathbb{N} is one-one but not onto).
- If f and g are both one-one, then $g \circ f$ is one-one; if both are onto, then $g \circ f$ is onto.
- The inverse of an invertible function is unique, and $(f^{-1})^{-1} = f$.
- Intersection of two equivalence relations is again an equivalence relation (but their union need not be).

Solved Examples

Example 1: Show that $R = \{(a, b) : a - b \text{ is divisible by } 2\}$ on the set of integers \mathbb{Z} is an equivalence relation.

Solution

Reflexive: $a - a = 0$ is divisible by 2, so (a, a) is in R .

Symmetric: if 2 divides $a - b$, then 2 divides $b - a$, so (b, a) is in R .

Transitive: if 2 divides $a - b$ and 2 divides $b - c$, then 2 divides $(a - b) + (b - c) = a - c$.

All three hold, so R is an equivalence relation. It produces two classes: even integers $[0]$ and odd integers $[1]$.

Example 2: Show that $f : \mathbb{N} \rightarrow \mathbb{N}$, $f(x) = 2x$, is one-one but not onto.

Solution

One-one: $f(x_1) = f(x_2)$ gives $2x_1 = 2x_2$, hence $x_1 = x_2$.

Not onto: the number 1 in the co-domain \mathbb{N} has no pre-image, because $2x = 1$ has no natural-number solution.

So f is injective but not surjective.

Example 3: Let $f(x) = 4x + 3$ from \mathbb{N} to $Y = \{y \in \mathbb{N} : y = 4x + 3\}$. Show f is invertible and find f^{-1} .

Solution

For any y in Y , $y = 4x + 3$ gives $x = (y - 3)/4$. Define $g(y) = (y - 3)/4$.

Then $g(f(x)) = (4x + 3 - 3)/4 = x$ and $f(g(y)) = 4 * (y - 3)/4 + 3 = y$.

Both composites are identities, so f is invertible and $f^{-1}(y) = (y - 3)/4$.

Common Mistakes

Avoid these errors

- Confusing co-domain with range. A function is onto only when range EQUALS co-domain, not just a subset.
- Assuming a reflexive-and-symmetric relation is automatically transitive — it is not.
- Writing $f \circ g$ when you mean $g \circ f$. Always apply the inner function first.
- Trying to invert a function that is not bijective; only one-one AND onto functions have inverses.

Exam Tips

Score-boosting tips

- To test one-one quickly, set $f(x_1) = f(x_2)$ and try to force $x_1 = x_2$. To test onto, pick an arbitrary y and solve $f(x) = y$ for x in the domain.
- For 'prove equivalence relation' questions, always check the three properties in the fixed order R-S-T and state each clearly for full marks.
- On a finite set, one-one and onto are equivalent — use this shortcut to save time in MCQs.

Chapter Summary

- A relation on A is any subset of $A \times A$; a function sends each input to exactly one output.
- Reflexive + Symmetric + Transitive = Equivalence relation, which partitions the set into classes.
- One-one (injective) + Onto (surjective) = Bijective = Invertible.
- Composition $(g \circ f)(x) = g(f(x))$ is associative but not commutative.
- An invertible function has a unique inverse f^{-1} that undoes it.

Quick Revision Sheet

Relations and Functions - 60-second recap

- Relation = subset of $A \times A$. Function = single output per input.
- Equivalence \Rightarrow splits set into disjoint classes.
- Injective: distinct inputs \rightarrow distinct outputs. Surjective: range = co-domain.
- Finite set: one-one \Leftrightarrow onto.
- Invertible \Leftrightarrow bijective; inverse is unique.

CHAPTER 2

Inverse Trigonometric Functions

Introduction

Trigonometric functions like \sin , \cos and \tan are periodic, so they repeat their values endlessly and cannot be reversed as they stand. By restricting each one to a carefully chosen interval where it becomes one-one and onto, we can define an inverse. These inverse trigonometric functions take a ratio and return the angle that produced it — and they appear constantly in calculus, physics and engineering.

Why This Chapter Matters

Inverse trig functions let us recover angles from measured ratios, which is essential in navigation, signal processing and mechanics. In calculus they appear as the antiderivatives of many algebraic expressions, so mastering their domains and ranges is non-negotiable for integration.

Real-Life Applications

- A ramp of known height and base length: the angle of inclination is $\arctan(\text{height} / \text{base})$.
- Robotics and animation use inverse trig to compute joint angles from end-effector positions (inverse kinematics).
- Finding the angle of elevation of the Sun from the length of a shadow uses \arctan .

Key Concepts

Principal value branch

To invert a trig function we keep only one cycle on which it is one-one. The output interval chosen for that cycle is the principal value branch, and the answer it gives is the principal value.

Domains and ranges

Each inverse trig function has a fixed domain (the allowed inputs) and a fixed principal range (the allowed angle outputs). These must be memorised exactly — most errors come from forgetting them.

Graphs

The graph of an inverse function is the mirror image of the original across the line $y = x$. So $y = \sin^{-1} x$ is the reflection of the principal piece of $y = \sin x$.

$\sin^{-1} x$ is not $(\sin x)^{-1}$

The superscript -1 here means 'inverse function', NOT reciprocal. $(\sin x)^{-1} = 1/\sin x = \operatorname{cosec} x$, which is completely different.

Definitions

arc sine
 $y = \sin^{-1} x$ means $\sin y = x$ with y in $[-\pi/2, \pi/2]$.

arc cosine
 $y = \cos^{-1} x$ means $\cos y = x$ with y in $[0, \pi]$.

arc tangent
 $y = \tan^{-1} x$ means $\tan y = x$ with y in $(-\pi/2, \pi/2)$.

Principal value

The unique value of an inverse trig function lying inside its principal range.

Formula / Key Results Table

$\sin^{-1} x$	domain $[-1, 1]$, range $[-\pi/2, \pi/2]$
$\cos^{-1} x$	domain $[-1, 1]$, range $[0, \pi]$
$\tan^{-1} x$	domain \mathbb{R} , range $(-\pi/2, \pi/2)$
$\operatorname{cosec}^{-1} x$	domain $\mathbb{R} - (-1, 1)$, range $[-\pi/2, \pi/2] - \{0\}$
$\sec^{-1} x$	domain $\mathbb{R} - (-1, 1)$, range $[0, \pi] - \{\pi/2\}$
$\cot^{-1} x$	domain \mathbb{R} , range $(0, \pi)$
Complementary pair	$\sin^{-1} x + \cos^{-1} x = \pi/2$
Complementary pair	$\tan^{-1} x + \cot^{-1} x = \pi/2$
Complementary pair	$\sec^{-1} x + \operatorname{cosec}^{-1} x = \pi/2$
Reciprocal	$\sin^{-1}(1/x) = \operatorname{cosec}^{-1} x$, for $ x \geq 1$
Negative argument	$\sin^{-1}(-x) = -\sin^{-1} x$; $\tan^{-1}(-x) = -\tan^{-1} x$
Negative argument	$\cos^{-1}(-x) = \pi - \cos^{-1} x$; $\cot^{-1}(-x) = \pi - \cot^{-1} x$
Sum ($xy < 1$)	$\tan^{-1} x + \tan^{-1} y = \tan^{-1}((x + y)/(1 - xy))$
Difference	$\tan^{-1} x - \tan^{-1} y = \tan^{-1}((x - y)/(1 + xy))$
Double	$2 \tan^{-1} x = \tan^{-1}(2x/(1 - x^2))$, $ x < 1$
Double	$2 \tan^{-1} x = \sin^{-1}(2x/(1 + x^2)) = \cos^{-1}((1 - x^2)/(1 + x^2))$

Important Properties

- $\sin(\sin^{-1} x) = x$ for x in $[-1, 1]$, but $\sin^{-1}(\sin x) = x$ only when x lies in $[-\pi/2, \pi/2]$.
- If the angle is outside the principal range, reduce it first using identities before applying the inverse function.
- The complementary identities (sum = $\pi/2$) are the fastest way to simplify mixed expressions.

Solved Examples

Example 1: Find the principal value of $\sin^{-1}(1/2)$.

Solution

We need the angle y in $[-\pi/2, \pi/2]$ with $\sin y = 1/2$.

$\sin(\pi/6) = 1/2$ and $\pi/6$ lies in the principal range.

So $\sin^{-1}(1/2) = \pi/6$.

Example 2: Find the principal value of $\cot^{-1}(-1/\sqrt{3})$.

Solution

We need y in $(0, \pi)$ with $\cot y = -1/\sqrt{3}$.
 $\cot(2\pi/3) = -1/\sqrt{3}$ and $2\pi/3$ is in $(0, \pi)$.
 So $\cot^{-1}(-1/\sqrt{3}) = 2\pi/3$.

Example 3: Evaluate $\sin^{-1}(\sin(3\pi/5))$.

Solution

$3\pi/5$ is NOT in $[-\pi/2, \pi/2]$, so we cannot cancel directly.
 Use $\sin(3\pi/5) = \sin(\pi - 3\pi/5) = \sin(2\pi/5)$, and $2\pi/5$ IS in the principal range.
 Therefore $\sin^{-1}(\sin(3\pi/5)) = 2\pi/5$.

Common Mistakes

Avoid these errors

- Reading $\sin^{-1} x$ as $1/\sin x$. It is the inverse function, not the reciprocal.
- Cancelling $\sin^{-1}(\sin x) = x$ without checking that x is inside $[-\pi/2, \pi/2]$.
- Using the $\tan^{-1} x + \tan^{-1} y$ sum formula when $xy \geq 1$ — an adjustment of $+\pi$ or $-\pi$ is then needed.
- Forgetting that \cos^{-1} outputs lie in $[0, \pi]$, so its principal value is never negative.

Exam Tips

Score-boosting tips

- Memorise the domain-range table as a single block — it is the source of almost all marks in this chapter.
- When an angle is outside the principal range, pull it back using $\sin(\pi - x)$, $\cos(2\pi - x)$, $\tan(x - \pi)$ etc.
- Substitution $x = \tan(\theta)$ or $x = \sin(\theta)$ turns ugly algebraic expressions into clean angle expressions.

Chapter Summary

- Inverse trig functions are defined only on restricted (principal) branches where the original is bijective.
- Know all six domain-range pairs precisely.
- Complementary identities: each co-function pair sums to $\pi/2$.
- $\sin^{-1}(\sin x) = x$ only on the principal branch; otherwise reduce the angle first.
- The -1 superscript means inverse, never reciprocal.

Quick Revision Sheet

Inverse Trigonometric Functions - 60-second recap

- $\sin^{-1}: [-1, 1] \rightarrow [-\pi/2, \pi/2]$. $\cos^{-1}: [-1, 1] \rightarrow [0, \pi]$. $\tan^{-1}: \mathbb{R} \rightarrow (-\pi/2, \pi/2)$.
- $\sin^{-1} x + \cos^{-1} x = \pi/2$ (and the other two co-pairs).
- $\tan^{-1} x + \tan^{-1} y = \tan^{-1}((x+y)/(1-xy))$ when $xy < 1$.
- $\cos^{-1}(-x) = \pi - \cos^{-1} x$.
- Always check the principal range before cancelling.

CHAPTER 3

Matrices

Introduction

A matrix is a rectangular grid of numbers arranged in rows and columns. It is one of the most powerful organising tools in mathematics: a single matrix can store the coefficients of many equations, the transformation of an image, or a table of data, and matrix operations let us manipulate all of that information at once.

Why This Chapter Matters

Matrices are the engine behind computer graphics, machine learning, economics models, network analysis and the solution of large systems of linear equations. Every rotation, scaling and projection on a screen is a matrix multiplication.

Real-Life Applications

- Computer graphics: rotating, scaling or moving a 3D object is done by multiplying its coordinate matrix by a transformation matrix.
- Spreadsheets and data tables are matrices; operations like totalling sales across regions are matrix additions.
- Cryptography encodes messages by multiplying them with an invertible key matrix and decodes with its inverse.

Key Concepts

Order of a matrix

A matrix with m rows and n columns has order $m \times n$. The entry in row i and column j is written a_{ij} .

Types of matrices

Row matrix (1 row), column matrix (1 column), square matrix ($m = n$), diagonal matrix (off-diagonal entries 0), scalar matrix (diagonal matrix with equal diagonal entries), identity matrix I (scalar matrix of 1s), zero matrix (all entries 0).

Equality of matrices

Two matrices are equal only if they have the same order AND every corresponding entry is equal.

Operations

Addition and subtraction are entrywise and need equal orders. Scalar multiplication multiplies every entry. Matrix multiplication AB is defined only when the number of columns of A equals the number of rows of B .

Transpose

The transpose A' (or A^T) is obtained by turning rows into columns. An $m \times n$ matrix becomes $n \times m$.

Symmetric and skew-symmetric

A is symmetric if $A' = A$; skew-symmetric if $A' = -A$ (its diagonal entries are then all 0). Every square matrix splits uniquely as the sum of a symmetric and a skew-symmetric part.

Definitions

Diagonal matrix

Square matrix with $a_{ij} = 0$ whenever i is not equal to j .

Identity matrix I	Diagonal matrix with every diagonal entry equal to 1.
Transpose A'	Matrix obtained by interchanging rows and columns of A.
Invertible matrix	Square matrix A for which there exists B with $AB = BA = I$.

Formula / Key Results Table

Order of product AB	$(m \times n)$ times $(n \times p)$ gives $(m \times p)$
Entry of product	$c_{ik} = \sum_j a_{ij} * b_{jk}$
Commutativity of addition	$A + B = B + A$
Distributive law	$A(B + C) = AB + AC$
Transpose of sum	$(A + B)' = A' + B'$
Transpose of product	$(AB)' = B' A'$ (order reverses)
Transpose of scalar multiple	$(kA)' = k A'$
Symmetric part	$(1/2)(A + A')$
Skew-symmetric part	$(1/2)(A - A')$
Inverse of a product	$(AB)^{-1} = B^{-1} A^{-1}$

Important Properties

- Matrix multiplication is associative and distributive but NOT commutative: in general AB is not equal to BA.
- AB can be a zero matrix even when neither A nor B is zero.
- $(A')' = A$, and the inverse of a square matrix, if it exists, is unique.
- Any square matrix = symmetric part + skew-symmetric part, and this decomposition is unique.

Solved Examples

Example 1: If A is a 2x3 matrix and B is a 3x2 matrix, what are the orders of AB and BA?

Solution

AB: columns of A (3) match rows of B (3), result has order 2 x 2.

BA: columns of B (2) match rows of A (2), result has order 3 x 3.

So AB and BA exist but have different orders — another reason multiplication is not commutative.

Example 2: Express $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Solution

$A' = \begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix}$.

Symmetric part $P = (1/2)(A + A') = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$.

Skew part $Q = (1/2)(A - A') = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Check: $P + Q = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = A$. Done.

Common Mistakes

Avoid these errors

- Trying to multiply matrices whose inner dimensions do not match.
- Assuming $AB = BA$ — order matters in matrix multiplication.
- Writing $(AB)' = A'B'$ instead of the correct $(AB)' = B'A'$.
- Forgetting that a skew-symmetric matrix must have all diagonal entries zero.

Exam Tips

Score-boosting tips

- Before multiplying, write the orders side by side: $(m \times n)(n \times p)$. The inner numbers must match; the outer numbers give the answer's order.
- For symmetric/skew decomposition, always compute A' first, then add and subtract.
- Use the identity matrix as a sanity check: $AI = IA = A$ for any compatible square A .

Chapter Summary

- A matrix is an $m \times n$ array; equality requires same order and equal entries.
- Add/subtract entrywise; multiply rows-into-columns when dimensions agree.
- Multiplication is associative and distributive but not commutative.
- Transpose reverses order in products: $(AB)' = B'A'$.
- Every square matrix = unique symmetric + skew-symmetric parts.

Quick Revision Sheet

Matrices - 60-second recap

- Order $m \times n$; entry a_{ij} .
- AB defined when $\text{cols}(A) = \text{rows}(B)$; result $m \times p$.
- $(AB)' = B'A'$, $(AB)^{-1} = B^{-1}A^{-1}$.
- Symmetric $A' = A$; skew $A' = -A$ (zero diagonal).
- $AB = O$ does not imply $A = O$ or $B = O$.

CHAPTER 4

Determinants

Introduction

Every square matrix has a single number attached to it called its determinant. This number tells us whether the matrix can be inverted, the area or volume scaling it produces, and whether a system of linear equations has a unique solution. Determinants connect the algebra of matrices with the geometry of space.

Why This Chapter Matters

The determinant is the gatekeeper of invertibility: a matrix is invertible exactly when its determinant is non-zero. This single test decides whether a linear system has one solution, none, or infinitely many — vital in engineering, economics and physics.

Real-Life Applications

- Solving simultaneous equations for currents in an electric circuit (Cramer's rule) relies on determinants.
- Computer graphics use the determinant to detect when a transformation flips or flattens a shape.
- Determinants give the area of a land plot directly from the coordinates of its corners.

Key Concepts

Determinant of order 2 and 3

For a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the determinant is $ad - bc$. For 3x3 we expand along any row or column using cofactors.

Minors and cofactors

The minor M_{ij} is the determinant left after deleting row i and column j . The cofactor is $A_{ij} = (-1)^{(i+j)} M_{ij}$ — a signed minor.

Adjoint

The adjoint of A is the transpose of the matrix of cofactors. It satisfies $A (\text{adj } A) = (\text{adj } A) A = |A| I$.

Inverse via determinant

If $|A|$ is not zero, then $A^{-1} = (1/|A|) \text{adj } A$. If $|A| = 0$ the matrix is singular and has no inverse.

Singular vs non-singular

Singular means $|A| = 0$ (not invertible). Non-singular means $|A| \neq 0$ (invertible).

Definitions

Minor M_{ij}	Determinant after deleting row i and column j .
Cofactor A_{ij}	$(-1)^{(i+j)}$ times the minor M_{ij} .
Adjoint $\text{adj } A$	Transpose of the cofactor matrix of A .
Singular matrix	Square matrix with determinant 0.

Formula / Key Results Table

2x2 determinant	$[[[a,b],[c,d]]] = ad - bc$
Expansion (3x3 along row 1)	$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$
Cofactor	$A_{ij} = (-1)^{(i+j)} M_{ij}$
Adjoint property	$A (\text{adj } A) = A I$
Inverse	$A^{-1} = (1/ A) \text{adj } A, A \text{ not } 0$
Determinant of product	$ AB = A B $
Determinant of transpose	$ A' = A $
Scalar multiple (order n)	$ kA = k^n A $
Determinant of inverse	$ A^{-1} = 1/ A $
Adjoint determinant (order n)	$ \text{adj } A = A ^{(n-1)}$
Area of triangle	$(1/2) x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) $

Important Properties

- Swapping two rows (or columns) changes the sign of the determinant.
- If any two rows (or columns) are identical or proportional, the determinant is 0.
- Multiplying one row by a scalar k multiplies the whole determinant by k .
- Adding a multiple of one row to another leaves the determinant unchanged — the basis of fast simplification.

Solved Examples

Example 1: Find the area of the triangle with vertices (1, 0), (6, 0) and (4, 3).

Solution

$$\begin{aligned} \text{Area} &= (1/2) |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| \\ &= (1/2) |1(0 - 3) + 6(3 - 0) + 4(0 - 0)| = (1/2) |-3 + 18 + 0| \\ &= (1/2)(15) = 7.5 \text{ square units.} \end{aligned}$$

Example 2: For $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, find A^{-1} .

Solution

$$\begin{aligned} |A| &= (2)(4) - (3)(1) = 5, \text{ which is non-zero, so } A \text{ is invertible.} \\ \text{adj } A &= \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} \text{ (swap diagonal, negate off-diagonal).} \\ A^{-1} &= (1/5) \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4/5 & -3/5 \\ -1/5 & 2/5 \end{bmatrix}. \end{aligned}$$

Common Mistakes

Avoid these errors

- Forgetting the alternating sign $(-1)^{(i+j)}$ when turning minors into cofactors.
- Computing the adjoint without transposing the cofactor matrix.
- Trying to find an inverse when $|A| = 0$ — singular matrices have none.
- Using $|kA| = k|A|$; for an $n \times n$ matrix it is $k^n |A|$.

Exam Tips

Score-boosting tips

- Expand along the row or column with the most zeros to minimise work.
- Use row operations (adding multiples of rows) to create zeros before expanding.
- Check invertibility with the determinant FIRST; only then compute the adjoint.

Chapter Summary

- The determinant is one number per square matrix; $|A| = 0$ means singular (non-invertible).
- Cofactor $A_{ij} = (-1)^{i+j} M_{ij}$; adjoint is the transpose of the cofactor matrix.
- $A^{-1} = (1/|A|) \text{adj } A$ when $|A|$ is non-zero.
- $|AB| = |A||B|$, $|A'| = |A|$, $|kA| = k^n |A|$.
- Determinants give triangle area and solve linear systems.

Quick Revision Sheet

Determinants - 60-second recap

- 2x2: $ad - bc$. 3x3: expand by cofactors.
- Cofactor sign pattern: $+$ - $+$ / - $+$ - / $+$ - $+$.
- $A (\text{adj } A) = |A| I$; $A^{-1} = \text{adj } A / |A|$.
- $|AB| = |A||B|$; $|kA| = k^n |A|$.
- Equal/proportional rows \Rightarrow determinant 0.

CHAPTER 5

Continuity and Differentiability

Introduction

Continuity captures the idea of a curve with no breaks, jumps or holes — you can draw it without lifting your pen. Differentiability is stronger: it means the curve is smooth enough to have a well-defined tangent (slope) at a point. This chapter formalises both ideas and builds the complete toolkit for differentiating almost any function you will meet.

Why This Chapter Matters

Differentiation is the mathematics of change — velocity, growth rate, marginal cost, current. Before you can apply derivatives you must know exactly when a function is continuous and differentiable, and how to differentiate composite, implicit, logarithmic and parametric functions.

Real-Life Applications

- Velocity is the derivative of position; acceleration is the derivative of velocity.
- Economists differentiate cost and revenue functions to find marginal cost and marginal revenue.
- Continuity guarantees that physical quantities like temperature change smoothly rather than teleporting between values.

Key Concepts

Continuity at a point

f is continuous at $x = c$ if the left-hand limit, the right-hand limit and the actual value $f(c)$ all exist and are equal. A function is continuous if it is continuous at every point of its domain.

Algebra of continuous functions

Sums, differences, products and quotients (where the denominator is non-zero) of continuous functions are continuous; so is the composite of continuous functions.

Differentiability

f is differentiable at c if the derivative (limit of the difference quotient) exists there. Every differentiable function is continuous, but a continuous function need not be differentiable (e.g. $|x|$ at $x = 0$).

Chain rule

To differentiate a composite $f(g(x))$, multiply the outer derivative by the inner derivative: $d/dx f(g(x)) = f'(g(x)) * g'(x)$.

Implicit differentiation

When y is tangled with x in one equation, differentiate every term with respect to x , treating y as a function of x , then solve for dy/dx .

Logarithmic differentiation

For functions of the form $[u(x)]^{v(x)}$ (a variable base AND a variable power), take the natural log of both sides first, then differentiate.

Definitions

Continuous at c	left limit = right limit = $f(c)$.
Derivative $f'(x)$	Instantaneous rate of change; slope of the tangent line.
Composite function	A function inside another, like $\sin(x^2)$.
Second derivative	The derivative of the derivative, written $f''(x)$ or d^2y/dx^2 .

Formula / Key Results Table

$d/dx (x^n)$	$n x^{(n-1)}$
$d/dx (\sin x)$	$\cos x$
$d/dx (\cos x)$	$-\sin x$
$d/dx (\tan x)$	$\sec^2 x$
$d/dx (e^x)$	e^x
$d/dx (\log x)$	$1/x$
$d/dx (a^x)$	$a^x \log a$
$d/dx (\sin^{-1} x)$	$1/\sqrt{1 - x^2}$
$d/dx (\cos^{-1} x)$	$-1/\sqrt{1 - x^2}$
$d/dx (\tan^{-1} x)$	$1/(1 + x^2)$
Product rule	$(uv)' = u'v + uv'$
Quotient rule	$(u/v)' = (u'v - uv')/v^2$
Chain rule	$dy/dx = (dy/dt)(dt/dx)$
Parametric form	$dy/dx = (dy/dt)/(dx/dt)$

Important Properties

- Differentiable implies continuous; the converse is false ($|x|$ is continuous but not differentiable at 0).
- A function can fail to be differentiable at sharp corners, cusps, vertical tangents or breaks.
- Polynomial, exponential, sine and cosine functions are differentiable everywhere.

Solved Examples

Example 1: Differentiate $y = \sin(x^2)$ with respect to x .

Solution

Outer function $\sin(\)$, inner function x^2 .

By the chain rule, $dy/dx = \cos(x^2) * d/dx(x^2)$.

$= \cos(x^2) * 2x = 2x \cos(x^2)$.

Example 2: Find dy/dx if $y = x^x$ ($x > 0$).

Solution

Take natural log: $\log y = x \log x$.

Differentiate both sides: $(1/y) dy/dx = \log x + x^*(1/x) = \log x + 1$.

So $dy/dx = y(1 + \log x) = x^x(1 + \log x)$.

Example 3: Check continuity of $f(x) = |x|$ at $x = 0$.

Solution

Left limit: as $x \rightarrow 0$ from below, $|x| \rightarrow 0$. Right limit: as $x \rightarrow 0$ from above, $|x| \rightarrow 0$.

$f(0) = 0$. All three agree, so f is continuous at 0.

However the left derivative (-1) and right derivative (+1) differ, so f is NOT differentiable at 0.

Common Mistakes

Avoid these errors

- Assuming continuity implies differentiability — it does not (corners break differentiability).
- Forgetting the inner derivative in the chain rule.
- Using the power rule on x^x ; a variable base AND variable power need logarithmic differentiation.
- In implicit differentiation, treating y as a constant instead of a function of x (you must include dy/dx).

Exam Tips

Score-boosting tips

- For continuity questions, always compute LHL, RHL and $f(c)$ separately and compare.
- Identify the outermost function first when applying the chain rule, then work inward.
- Any expression with the variable in both base and exponent is a signal to use logarithms.

Chapter Summary

- Continuous at c means $LHL = RHL = f(c)$.
- Differentiable \Rightarrow continuous, but not the other way round.
- Chain rule handles composites; product and quotient rules handle combinations.
- Implicit differentiation solves for dy/dx when y is not isolated.
- Logarithmic differentiation handles $[u(x)]^{v(x)}$.

Quick Revision Sheet

Continuity and Differentiability - 60-second recap

- Continuity: $LHL = RHL = f(c)$.
- $d/dx \sin^{-1} x = 1/\sqrt{1-x^2}$; $d/dx \tan^{-1} x = 1/(1+x^2)$.
- Chain: $dy/dx = (dy/dt)(dt/dx)$.
- Parametric: $dy/dx = (dy/dt)/(dx/dt)$.
- $x^x \Rightarrow$ take log first.

CHAPTER 6

Application of Derivatives

Introduction

Once we can differentiate, the derivative becomes a measuring instrument. It tells us how fast quantities change, whether a function is climbing or falling, where its highest and lowest points lie, and the slope of tangents and normals to curves. This chapter turns the abstract derivative into practical problem-solving power.

Why This Chapter Matters

Optimisation — getting the maximum profit, minimum cost, largest volume or shortest time — is one of the most useful skills in all of applied mathematics, and it is done with derivatives. Rates of change describe motion, growth and decay everywhere in science.

Real-Life Applications

- A company finds the production level that maximises profit by setting the derivative of the profit function to zero.
- Engineers design a container of fixed material to hold the maximum volume — a classic maxima problem.
- Physicists relate the rate at which a balloon's radius grows to the rate at which its volume grows (related rates).

Key Concepts

Rate of change

dy/dx is the rate of change of y with respect to x . If both depend on time t , the chain rule links their rates: $dy/dt = (dy/dx)(dx/dt)$.

Increasing and decreasing functions

On an interval, f is increasing where $f'(x) \geq 0$ and decreasing where $f'(x) \leq 0$. The sign of the first derivative tells you the direction of the curve.

Critical points

A critical point is where $f'(x) = 0$ or f' does not exist. Maxima and minima can only occur at critical points or at the ends of the interval.

First derivative test

At a critical point c : if f' changes from $+$ to $-$, c is a local maximum; from $-$ to $+$, a local minimum; if it does not change sign, c is a point of inflexion.

Second derivative test

If $f'(c) = 0$ and $f''(c) < 0$, then c is a local maximum; if $f''(c) > 0$, a local minimum; if $f''(c) = 0$ the test is inconclusive.

Tangents and normals

The slope of the tangent at (x_0, y_0) is $f'(x_0)$. The normal is perpendicular, with slope $-1/f'(x_0)$.

Definitions

Local maximum	A point higher than all nearby points.
Local minimum	A point lower than all nearby points.
Critical point	Where $f'(x) = 0$ or f' is undefined.
Point of inflexion	A point where the curve changes its bending direction.

Formula / Key Results Table

Rate via chain rule	$dy/dt = (dy/dx)(dx/dt)$
Increasing	$f'(x) \geq 0$ on the interval
Decreasing	$f'(x) \leq 0$ on the interval
Tangent slope at x_0	$m = f'(x_0)$
Tangent line	$y - y_0 = f'(x_0)(x - x_0)$
Normal slope	$-1/f'(x_0)$
First derivative test	sign change of f' at c gives max/min
Second derivative test	$f'(c)=0, f''(c)<0 \rightarrow$ max; $f''(c)>0 \rightarrow$ min

Important Properties

- Absolute (global) maximum and minimum on a closed interval occur either at critical points or at the endpoints — always test both.
- If $f'(x) = 0$ throughout an interval, f is constant there.
- A function may have several local maxima and minima but only one absolute maximum and one absolute minimum on a closed interval.

Solved Examples

Example 1: Find the local maximum and minimum of $f(x) = x^3 - 3x$.

Solution

$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$. Critical points: $x = 1$ and $x = -1$.

$f''(x) = 6x$. At $x = -1$, $f'' = -6 < 0 \rightarrow$ local maximum, value $f(-1) = 2$.

At $x = 1$, $f'' = 6 > 0 \rightarrow$ local minimum, value $f(1) = -2$.

Example 2: A balloon's radius increases at 2 cm/s. How fast is the volume increasing when $r = 5$ cm?

Solution

Volume $V = (4/3) \pi r^3$, so $dV/dr = 4 \pi r^2$.

$dV/dt = (dV/dr)(dr/dt) = 4 \pi r^2 * 2 = 8 \pi r^2$.

At $r = 5$: $dV/dt = 8 \pi (25) = 200 \pi \text{ cm}^3/\text{s}$.

Common Mistakes

Avoid these errors

- Reporting a critical point as a maximum/minimum without confirming with a derivative test.
- Ignoring the endpoints when finding the absolute maximum or minimum on a closed interval.
- Confusing the normal's slope with the tangent's; the normal slope is $-1/f'(x_0)$.
- Forgetting that $f''(c) = 0$ makes the second derivative test inconclusive — fall back to the first derivative test.

Exam Tips

Score-boosting tips

- For optimisation word problems: write the quantity to be optimised as a function of ONE variable using the given constraint, then differentiate.
- Always list endpoints alongside critical points when an interval is closed.
- Use the second derivative test for speed, but switch to the first derivative test when $f''(c) = 0$.

Chapter Summary

- f' measures rate of change; its sign shows increasing/decreasing behaviour.
- Maxima and minima occur at critical points or interval endpoints.
- First derivative test: sign change of f' . Second derivative test: sign of f'' .
- Tangent slope = $f'(x_0)$; normal slope = $-1/f'(x_0)$.
- Optimisation reduces a real problem to maximising or minimising one function.

Quick Revision Sheet

Application of Derivatives - 60-second recap

- $f'(x) > 0$ increasing; $f'(x) < 0$ decreasing.
- Critical point: $f'(x) = 0$ or undefined.
- Max: $f'' < 0$; Min: $f'' > 0$.
- Normal slope = $-1/(\text{tangent slope})$.
- Closed interval: also test endpoints for absolute extrema.

CHAPTER 7

Proofs in Mathematics

Introduction

A proof is a watertight chain of logical steps that establishes a mathematical statement beyond doubt. Mathematics does not run on examples or intuition alone — a result is accepted only when it is proved. This appendix surveys the standard proof techniques every serious student should be able to recognise and use.

Why This Chapter Matters

Proof writing trains rigorous, structured thinking. In examinations, 'show that' and 'prove that' questions carry serious marks and demand a clear method. Beyond exams, the ability to justify a claim logically is valuable in coding, law, science and everyday reasoning.

Real-Life Applications

- Software verification uses proof techniques to guarantee that a program meets its specification.
- Mathematical induction underlies the analysis of algorithms and recursive programs.
- Proof by contradiction is the standard way to show that something is impossible (e.g. $\sqrt{2}$ is irrational).

Key Concepts

Direct proof

Start from the given hypothesis and move step by logical step, using definitions and known theorems, straight to the conclusion.

Proof by mathematical induction

Prove a statement for the base case (usually $n = 1$), then assume it for $n = k$ and prove it for $n = k + 1$. This establishes it for all natural numbers.

Proof by contradiction

Assume the statement is FALSE, then derive a logical impossibility. The contradiction forces the original statement to be true.

Proof by contrapositive

To prove 'if p then q ', instead prove the logically equivalent 'if not q then not p '.

Disproof by counter-example

A single example where the statement fails is enough to disprove a universal claim.

Definitions

Theorem	A statement that has been proved true.
Conjecture	A statement believed true but not yet proved.
Contrapositive	'If not q then not p ', equivalent to 'if p then q '.

Counter-example

A specific case that shows a general claim is false.

Formula / Key Results Table

Direct	$p \Rightarrow r \Rightarrow s \Rightarrow \dots \Rightarrow q$
Induction	Base case + $(S(k) \Rightarrow S(k+1))$ gives $S(n)$ for all n
Contradiction	Assume not- q , reach an impossibility
Contrapositive	$(p \Rightarrow q)$ is equivalent to $(\text{not } q \Rightarrow \text{not } p)$

Important Properties

- A statement and its contrapositive are always logically equivalent; a statement and its converse are NOT.
- One counter-example destroys a universal statement, but no number of examples can prove one.
- Induction needs BOTH the base step and the inductive step — neither alone is sufficient.

Solved Examples

Example 1: Prove (direct) that $f(x) = 2x + 5$ from \mathbb{R} to \mathbb{R} is one-one.

Solution

Assume $f(x_1) = f(x_2)$, i.e. $2x_1 + 5 = 2x_2 + 5$.

Subtract 5: $2x_1 = 2x_2$. Divide by 2: $x_1 = x_2$.

Equal outputs force equal inputs, so f is one-one.

Example 2: Disprove: 'every odd number is prime'.

Solution

Consider the odd number 9.

$9 = 3 \times 3$, so it is composite, not prime.

This single counter-example disproves the statement.

Common Mistakes

Avoid these errors

- Confusing the converse with the contrapositive — only the contrapositive is equivalent to the original.
- Omitting the base case in an induction proof.
- Believing that listing many examples proves a general statement (it does not).
- In contradiction, forgetting to clearly state the assumption that the claim is false.

Exam Tips

Score-boosting tips

- Choose the technique to fit the statement: 'for all natural n ' suggests induction; 'is impossible / irrational' suggests contradiction.
- When a direct proof feels hard, try the contrapositive — it is often easier.
- To disprove, hunt for the simplest possible counter-example.

Chapter Summary

- A proof is a logical chain from hypothesis to conclusion.
- Main methods: direct, induction, contradiction, contrapositive.
- Contrapositive is equivalent to the original; converse is not.
- Induction = base case + inductive step.
- One counter-example disproves a universal claim.

Quick Revision Sheet

Proofs in Mathematics - 60-second recap

- Direct: hypothesis \rightarrow ... \rightarrow conclusion.
- Induction: prove $n=1$, then $k \Rightarrow k+1$.
- Contradiction: assume false, reach impossibility.
- Contrapositive: $\text{not } q \Rightarrow \text{not } p$.
- Counter-example disproves 'for all' statements.

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CHAPTER 8

Mathematical Modelling

Introduction

Mathematical modelling is the art of translating a real-world situation into mathematics, solving it, and translating the answer back into the real world. It is how mathematics earns its keep in science, engineering, economics and medicine — by giving usable predictions about problems that are hard or impossible to attack directly.

Why This Chapter Matters

Modelling is the bridge between classroom mathematics and real applications. It teaches you to make sensible assumptions, choose the right tools (calculus, matrices, optimisation) and judge whether an answer is realistic — skills prized in every quantitative profession.

Real-Life Applications

- Estimating the height of a tower or the width of a river without measuring it directly, using angles and trigonometry.
- Predicting population growth or the spread of a disease with difference and differential equations.
- Planning the cheapest transport schedule using linear programming and matrices.

Key Concepts

What a model is

A mathematical model is a description of a real system using equations, formulae, graphs or programs that captures its essential behaviour while ignoring unimportant detail.

The modelling cycle

Identify the real situation, convert it to mathematics with variables and laws, solve the mathematics, interpret the result back in reality, and compare with observation. If it disagrees, refine the assumptions and repeat.

Assumptions

Every model rests on simplifying assumptions (ignore air resistance, treat population growth as smooth, etc.). Good assumptions make the problem solvable without destroying its realism.

Validation and refinement

A model is only as good as its agreement with reality. We test it against data and improve the assumptions until predictions are acceptable.

Definitions

Model	A mathematical description of a real-world situation.
Assumption	A simplifying condition that makes the model tractable.
Validation	Checking model predictions against real observations.
Parameter	A quantity in the model that can be tuned to fit reality.

Formula / Key Results Table

Step 1	Identify the real-world situation
Step 2	Translate into mathematics (variables, laws, equations)
Step 3	Solve the mathematical problem
Step 4	Interpret and compare with reality
Step 5	Refine assumptions and repeat if needed

Important Properties

- A model is a simplification, never a perfect copy — its value lies in being useful, not exact.
- Different assumptions can lead to different valid models of the same situation.
- The same mathematical tool (e.g. a differential equation) can model wildly different real systems.

Solved Examples

Example 1: Model the height H of a tower observed at angle of elevation α from horizontal distance l , with eye height h .

Solution

Assume level ground and a straight line of sight.

From the right triangle, $\tan(\alpha) = (H - h)/l$.

Solve: $H = h + l \tan(\alpha)$. Measuring α and l now gives the tower's height without climbing it.

Example 2: Outline a model for the growth of a bank deposit at a fixed annual interest rate.

Solution

Assumption: interest is added at regular intervals and the rate r is constant.

Model: after n periods the amount is $A = P(1 + r)^n$, where P is the principal.

Interpretation: the deposit grows geometrically; doubling time can be read off the model.

Common Mistakes

Avoid these errors

- Skipping the assumptions step, which makes the model impossible to judge or improve.
- Forgetting to translate the mathematical answer back into the language of the original problem.
- Treating a model's output as exact truth rather than an approximation.
- Choosing variables or laws that do not actually govern the situation.

Exam Tips

Score-boosting tips

- Always state your assumptions explicitly — examiners reward this and it clarifies your thinking.
- Follow the five-step cycle in order; do not jump straight to formulae.
- After solving, ask 'is this answer physically reasonable?' before accepting it.

Chapter Summary

- Modelling converts a real problem into mathematics, solves it, and interprets the result.
- The cycle: identify -> translate -> solve -> interpret -> refine.
- Assumptions make models solvable; validation keeps them honest.
- A model is a useful simplification, never a perfect replica.
- The same mathematics can model many different real systems.

Quick Revision Sheet

Mathematical Modelling - 60-second recap

- Five steps: identify, translate, solve, interpret, refine.
- State assumptions explicitly.
- Validate against real data.
- Models are approximations, not exact truth.
- Tools: calculus, matrices, linear programming.

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CHAPTER 9

Formula Handbook

A compact, chapter-wise reference of every key formula and result in this volume. Use it for last-minute revision before the exam.

1. Relations and Functions

Relations on n-element set	$2^{(n^2)}$
Functions $A(m) \rightarrow B(n)$	n^m
One-one functions ($m \leq n$)	$n!/(n-m)!$
Composition	$(g \circ f)(x) = g(f(x))$
Invertible	bijective; inverse unique, $(f^{-1})^{-1} = f$

2. Inverse Trigonometric Functions

$\sin^{-1} / \cos^{-1} / \tan^{-1}$ ranges	$[-\pi/2, \pi/2] / [0, \pi] / (-\pi/2, \pi/2)$
Co-function sums	$\sin^{-1} x + \cos^{-1} x = \pi/2$ (and pairs)
tan sum	$\tan^{-1} x + \tan^{-1} y = \tan^{-1}((x+y)/(1-xy)), xy < 1$
Double angle	$2 \tan^{-1} x = \tan^{-1}(2x/(1-x^2))$
Negative	$\cos^{-1}(-x) = \pi - \cos^{-1} x$

3. Matrices

Product order	$(m \times n)(n \times p) = (m \times p)$
Transpose of product	$(AB)^T = B^T A^T$
Inverse of product	$(AB)^{-1} = B^{-1} A^{-1}$
Symmetric / skew parts	$(A+A^T)/2$ and $(A-A^T)/2$
Non-commutative	AB is generally not BA

4. Determinants

2x2	$ad - bc$
Cofactor	$A_{ij} = (-1)^{(i+j)} M_{ij}$
Inverse	$A^{-1} = \text{adj } A / A , A \neq 0$
Product / transpose	$ AB = A B , A^T = A $
Scalar / adjoint	$ kA = k^n A , \text{adj } A = A ^{(n-1)}$

5. Continuity and Differentiability

Continuity	$LHL = RHL = f(c)$
$d/dx \sin^{-1} x, \tan^{-1} x$	$1/\sqrt{1-x^2}, 1/(1+x^2)$
Product / quotient	$u'v+uv', (u'v-uv')/v^2$
Chain	$dy/dx = (dy/dt)(dt/dx)$
Parametric	$dy/dx = (dy/dt)/(dx/dt)$

6. Application of Derivatives

Increasing / decreasing	$f' \geq 0 / f' \leq 0$
Critical point	$f'(x) = 0$ or undefined
Second derivative test	$f'' < 0$ max, $f'' > 0$ min
Tangent / normal slope	$f'(x_0) / -1/f'(x_0)$
Related rates	$dy/dt = (dy/dx)(dx/dt)$

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CHAPTER 10
Quick Revision Notes

The fastest possible pass over all eight topics - ideal for the night before the exam. Each block is the 60-second recap for one chapter.

1. Relations and Functions

- Relation = subset of $A \times A$. Function = single output per input.
- Equivalence \Rightarrow splits set into disjoint classes.
- Injective: distinct inputs \rightarrow distinct outputs. Surjective: range = co-domain.
- Finite set: one-one \Leftrightarrow onto.
- Invertible \Leftrightarrow bijective; inverse is unique.

2. Inverse Trigonometric Functions

- $\sin^{-1}: [-1, 1] \rightarrow [-\pi/2, \pi/2]$. $\cos^{-1}: [-1, 1] \rightarrow [0, \pi]$. $\tan^{-1}: \mathbb{R} \rightarrow (-\pi/2, \pi/2)$.
- $\sin^{-1} x + \cos^{-1} x = \pi/2$ (and the other two co-pairs).
- $\tan^{-1} x + \tan^{-1} y = \tan^{-1}((x+y)/(1-xy))$ when $xy < 1$.
- $\cos^{-1}(-x) = \pi - \cos^{-1} x$.
- Always check the principal range before cancelling.

3. Matrices

- Order $m \times n$; entry a_{ij} .
- AB defined when $\text{cols}(A) = \text{rows}(B)$; result $m \times p$.
- $(AB)' = B'A'$, $(AB)^{-1} = B^{-1}A^{-1}$.
- Symmetric $A' = A$; skew $A' = -A$ (zero diagonal).
- $AB = O$ does not imply $A = O$ or $B = O$.

4. Determinants

- 2x2: $ad - bc$. 3x3: expand by cofactors.
- Cofactor sign pattern: $+ - + / - + - / + - +$.
- $A (\text{adj } A) = |A| I$; $A^{-1} = \text{adj } A / |A|$.
- $|AB| = |A||B|$; $|kA| = k^n |A|$.
- Equal/proportional rows \Rightarrow determinant 0.

5. Continuity and Differentiability

- Continuity: LHL = RHL = $f(c)$.
- $d/dx \sin^{-1} x = 1/\sqrt{1-x^2}$; $d/dx \tan^{-1} x = 1/(1+x^2)$.
- Chain: $dy/dx = (dy/dt)(dt/dx)$.
- Parametric: $dy/dx = (dy/dt)/(dx/dt)$.
- $x^x \Rightarrow$ take log first.

6. Application of Derivatives

- $f'(x) > 0$ increasing; $f'(x) < 0$ decreasing.
- Critical point: $f'(x) = 0$ or undefined.
- Max: $f'' < 0$; Min: $f'' > 0$.
- Normal slope = $-1/(\text{tangent slope})$.
- Closed interval: also test endpoints for absolute extrema.

7. Proofs in Mathematics

- Direct: hypothesis \rightarrow ... \rightarrow conclusion.
- Induction: prove $n=1$, then $k \Rightarrow k+1$.
- Contradiction: assume false, reach impossibility.
- Contrapositive: not $q \Rightarrow$ not p .
- Counter-example disproves 'for all' statements.

8. Mathematical Modelling

- Five steps: identify, translate, solve, interpret, refine.
- State assumptions explicitly.
- Validate against real data.
- Models are approximations, not exact truth.
- Tools: calculus, matrices, linear programming.

CHAPTER 11

Frequently Asked Questions

Twenty-five common doubts students raise about Part I, answered concisely.

Q1. What is the difference between a relation and a function?

A. Every function is a relation, but a function has the extra rule that each input maps to exactly one output. A general relation may link an input to many outputs.

Q2. How do I quickly check if a function is one-one?

A. Set $f(x_1) = f(x_2)$ and simplify. If this forces $x_1 = x_2$, the function is one-one (injective).

Q3. When is a function onto?

A. When its range equals its co-domain, i.e. every element of the co-domain has at least one pre-image.

Q4. Does every function have an inverse?

A. No. Only bijective functions (one-one AND onto) are invertible.

Q5. Why must inverse trig functions use restricted domains?

A. Trig functions are periodic and repeat values, so they are not one-one. Restricting them to a principal branch makes them invertible.

Q6. Is $\sin^{-1} x$ the same as $1/\sin x$?

A. No. $\sin^{-1} x$ is the inverse function (an angle). $1/\sin x$ is cosec x , a reciprocal.

Q7. What is the principal value?

A. The unique output of an inverse trig function that lies inside its agreed principal range.

Q8. Is matrix multiplication commutative?

A. No. In general AB is not equal to BA ; sometimes only one of them is even defined.

Q9. Can the product of two non-zero matrices be zero?

A. Yes. AB can be the zero matrix even when neither A nor B is zero.

Q10. What does the transpose do to a product?

A. It reverses the order: $(AB)' = B'A'$.

Q11. When is a matrix invertible?

A. Exactly when it is square and its determinant is non-zero (non-singular).

Q12. What does a zero determinant mean?

A. The matrix is singular: it has no inverse and the related linear system has no unique solution.

Q13. What is the adjoint of a matrix?

A. The transpose of its cofactor matrix; it satisfies $A(\text{adj } A) = |A| I$.

Q14. How are minors and cofactors related?

A. A cofactor is a signed minor: $A_{ij} = (-1)^{(i+j)} M_{ij}$.

Q15. Does continuity imply differentiability?

A. No. Differentiability implies continuity, but a continuous function can have corners (like $|x|$) where it is not differentiable.

Q16. When should I use logarithmic differentiation?

A. When the variable appears in both the base and the exponent, such as $y = x^x$.

Q17. What is the chain rule in one line?

A. Differentiate the outer function, then multiply by the derivative of the inner function.

Q18. How do I find dy/dx for parametric equations?

A. Compute dy/dt and dx/dt separately, then divide: $dy/dx = (dy/dt)/(dx/dt)$.

Q19. Where can maxima and minima occur?

A. Only at critical points (where $f' = 0$ or is undefined) or at the endpoints of a closed interval.

Q20. What is the difference between local and absolute extrema?

A. Local extrema are highest/lowest among nearby points; absolute extrema are highest/lowest over the whole interval.

Q21. When does the second derivative test fail?

A. When $f''(c) = 0$; then fall back to the first derivative test.

Q22. What is the slope of the normal to a curve?

A. The negative reciprocal of the tangent slope: $-1/f'(x_0)$.

Q23. What proof method suits 'true for all natural n '?

A. Mathematical induction: prove the base case, then show $n = k$ implies $n = k + 1$.

Q24. How do I disprove a general statement?

A. Find a single counter-example where the statement fails.

Q25. What are the steps of mathematical modelling?

A. Identify the situation, translate to mathematics, solve, interpret the result, and refine the assumptions if needed.

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CHAPTER 12

50 Most Probable Questions

Fifty exam-oriented questions spanning all topics, each with a short answer, a detailed explanation and a difficulty rating (Easy / Medium / Hard).

Q1. Show that the relation $R = \{(a,b): a-b \text{ is divisible by } 3\}$ on Z is an equivalence relation.

EASY

Answer: It is reflexive, symmetric and transitive.

Explanation: $3 \mid a-a = 0$ (reflexive); $3 \mid a-b$ implies $3 \mid b-a$ (symmetric); $3 \mid a-b$ and $3 \mid b-c$ imply $3 \mid a-c$ (transitive). Hence equivalence, with classes $[0],[1],[2]$.

Q2. Prove that $f: R \rightarrow R, f(x) = 3x + 2$ is a bijection.

EASY

Answer: It is one-one and onto.

Explanation: $f(x_1)=f(x_2)$ gives $3x_1+2 = 3x_2+2$ so $x_1 = x_2$ (one-one). For any $y, x = (y-2)/3$ gives $f(x) = y$ (onto). Hence bijective.

Q3. Find the number of one-one functions from a 3-element set to a 4-element set.

EASY

Answer: 24.

Explanation: Count = $n!/(n-m)! = 4!/(4-3)! = 24/1 = 24$.

Q4. If $f(x) = x^2$ on R , state whether it is one-one and onto.

EASY

Answer: Neither.

Explanation: $f(-1) = f(1) = 1$, so not one-one; negative numbers have no pre-image, so not onto.

Q5. Find $(g \circ f)(x)$ if $f(x) = \cos x$ and $g(x) = 3x^2$.

EASY

Answer: $3 \cos^2 x$.

Explanation: $(g \circ f)(x) = g(f(x)) = g(\cos x) = 3(\cos x)^2 = 3 \cos^2 x$.

Q6. Find the principal value of $\cos^{-1}(-1/2)$.

EASY

Answer: $2\pi/3$.

Explanation: \cos^{-1} outputs lie in $[0,\pi]$; $\cos(2\pi/3) = -1/2$, so the answer is $2\pi/3$.

Q7. Find the principal value of $\tan^{-1}(-1)$.

EASY

Answer: $-\pi/4$.

Explanation: \tan^{-1} range is $(-\pi/2, \pi/2)$; $\tan(-\pi/4) = -1$.

Q8. Evaluate $\cos^{-1}(\cos(7\pi/6))$.

MEDIUM

Answer: $5\pi/6$.

Explanation: $7\pi/6$ is outside $[0,\pi]$. $\cos(7\pi/6) = \cos(2\pi - 7\pi/6) = \cos(5\pi/6)$; $5\pi/6$ is in range.

Q9. Prove that $\sin^{-1} x + \cos^{-1} x = \pi/2$ for x in $[-1,1]$.

MEDIUM

Answer: Identity holds.

Explanation: Let $\sin^{-1} x = A$ so $\sin A = x$ and $\cos(\pi/2 - A) = x$; thus $\cos^{-1} x = \pi/2 - A$, giving the sum $\pi/2$.

Q10. Simplify $\tan^{-1}(1/2) + \tan^{-1}(1/3)$.

MEDIUM

Answer: $\pi/4$.

Explanation: Using $\tan^{-1} x + \tan^{-1} y = \tan^{-1}((x+y)/(1-xy)) = \tan^{-1}((5/6)/(5/6)) = \tan^{-1}(1) = \pi/4$.

Q11. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find A' .

EASY

Answer: $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

Explanation: The transpose swaps rows and columns: row i becomes column i .

Q12. Give the order of AB if A is 3x2 and B is 2x4.

EASY

Answer: 3x4.

Explanation: Inner dimensions (2,2) match; the result has the outer dimensions 3x4.

Q13. Express $A = \begin{bmatrix} 3 & 5 \\ 1 & 7 \end{bmatrix}$ as symmetric + skew-symmetric.

MEDIUM

Answer: Sym $\begin{bmatrix} 3 & 3 \\ 3 & 7 \end{bmatrix}$, Skew $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

Explanation: Sym = $(A+A')/2$, Skew = $(A-A')/2$; their sum returns A.

Q14. If A is skew-symmetric, what are its diagonal entries?

EASY

Answer: All zero.

Explanation: $A' = -A$ forces $a_{ii} = -a_{ii}$, so $2a_{ii} = 0$, hence $a_{ii} = 0$.

Q15. State the condition for $AB = BA$ to even be possible for both to exist and be equal in order.

MEDIUM

Answer: A and B must be square of the same order.

Explanation: Only same-order square matrices guarantee both products exist with matching order; equality still is not automatic.

Q16. Evaluate the determinant of $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$.

EASY

Answer: -2.

Explanation: $ad - bc = (2)(5) - (3)(4) = 10 - 12 = -2$.

Q17. Find $|A|$ for $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 5 \end{bmatrix}$.

MEDIUM

Answer: -9.

Explanation: Expand along row 2: only 3 survives, $|A| = 3 * \text{cofactor} = 3 * (+1)(1*5 - 2*4) = 3(5-8) = -9$.

Q18. For $A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$, find A^{-1} .

MEDIUM

Answer: $\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$.

Explanation: $|A| = 8 - 7 = 1$; $\text{adj } A = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$; $A^{-1} = \text{adj } A / 1$.

Q19. If $|A| = 5$ for a 3x3 matrix, find $|2A|$.

MEDIUM

Answer: 40.

Explanation: $|kA| = k^n |A| = 2^3 * 5 = 8 * 5 = 40$.

Q20. Find the area of the triangle with vertices (0,0), (4,0), (0,3).

EASY

Answer: 6 sq units.

Explanation: Area = $(1/2)|0(0-3)+4(3-0)+0(0-0)| = (1/2)(12) = 6$.

Q21. If $|\text{adj } A| = 49$ for a 3x3 matrix, find $|A|$.

HARD

Answer: 7 or -7.

Explanation: $|\text{adj } A| = |A|^{(n-1)} = |A|^2 = 49$, so $|A| = 7$ or -7 .

Q22. Solve using determinants whether $x+y=2$, $2x+2y=5$ has a unique solution.

MEDIUM

Answer: No unique solution.

Explanation: Coefficient determinant = $(1)(2)-(1)(2) = 0$, so the system is not uniquely solvable (here inconsistent).

Q23. Check continuity of $f(x) = x^2$ at $x = 2$.

EASY

Answer: Continuous.

Explanation: LHL = RHL = $4 = f(2)$; all equal, so continuous.

Q24. Is $f(x) = |x|$ differentiable at $x = 0$?

MEDIUM

Answer: No.

Explanation: Left derivative -1 , right derivative $+1$; they differ, so not differentiable (though it is continuous).

Q25. Differentiate $y = \sin(3x)$.

EASY

Answer: $3 \cos(3x)$.

Explanation: Chain rule: derivative of sin is cos, times derivative of $3x = 3$.

- Q26. Differentiate $y = e^{(x^2)}$.** EASY
Answer: $2x e^{(x^2)}$.
 Explanation: Chain rule: $e^{(x^2)}$ times derivative of $x^2 = 2x$.
- Q27. Find dy/dx for $y = x^x, x > 0$.** MEDIUM
Answer: $x^x (1 + \log x)$.
 Explanation: Take log: $\log y = x \log x$; differentiate: $(1/y)y' = \log x + 1$; so $y' = x^x(1+\log x)$.
- Q28. Find dy/dx if $x = a \cos t, y = a \sin t$.** MEDIUM
Answer: $-\cot t$.
 Explanation: $dx/dt = -a \sin t, dy/dt = a \cos t$; $dy/dx = (a \cos t)/(-a \sin t) = -\cot t$.
- Q29. Differentiate $y = \log(\sin x)$.** EASY
Answer: $\cot x$.
 Explanation: $(1/\sin x) * \cos x = \cos x / \sin x = \cot x$.
- Q30. Differentiate $y = \tan^{-1} x$ and state its value at $x = 1$.** EASY
Answer: $1/(1+x^2)$; value $1/2$.
 Explanation: Standard derivative $1/(1+x^2)$; at $x = 1$ it equals $1/2$.
- Q31. Find the second derivative of $y = x^4$.** EASY
Answer: $12x^2$.
 Explanation: $y' = 4x^3, y'' = 12x^2$.
- Q32. Differentiate implicitly: $x^2 + y^2 = 25$, find dy/dx .** MEDIUM
Answer: $-x/y$.
 Explanation: $2x + 2y dy/dx = 0$, so $dy/dx = -x/y$.
- Q33. For what x is $f(x) = x^2 - 4x$ increasing?** EASY
Answer: $x > 2$.
 Explanation: $f'(x) = 2x - 4 > 0$ gives $x > 2$.
- Q34. Find the critical points of $f(x) = x^3 - 6x^2 + 9x$.** MEDIUM
Answer: $x = 1$ and $x = 3$.
 Explanation: $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0$.
- Q35. Classify the critical points of $f(x) = x^3 - 3x$.** MEDIUM
Answer: $x=-1$ local max, $x=1$ local min.
 Explanation: $f'' = 6x$; at -1 it is negative (max, value 2), at 1 positive (min, value -2).
- Q36. Find the slope of the tangent to $y = x^2$ at $x = 3$.** EASY
Answer: 6.
 Explanation: $dy/dx = 2x$; at $x = 3$ slope = 6.
- Q37. Find the equation of the normal to $y = x^2$ at $(1,1)$.** MEDIUM
Answer: $x + 2y = 3$.
 Explanation: Tangent slope 2, normal slope $-1/2$; $y - 1 = -1/2 (x - 1)$ gives $x + 2y = 3$.
- Q38. A spherical balloon's radius grows at 2 cm/s; find dV/dt at $r = 5$.** MEDIUM
Answer: $200 \pi \text{ cm}^3/\text{s}$.
 Explanation: $V = (4/3)\pi r^3, dV/dt = 4 \pi r^2 (dr/dt) = 4 \pi (25)(2) = 200 \pi$.
- Q39. Find two positive numbers with sum 16 whose product is maximum.** MEDIUM
Answer: 8 and 8.
 Explanation: Let numbers be x and $16-x$; product $P = 16x - x^2$; $P' = 16 - 2x = 0$ gives $x = 8$.

Q40. Find the absolute maximum of $f(x) = 2x^3 - 3x^2$ on $[-1, 2]$.

HARD

Answer: 4 at $x = 2$.

Explanation: Critical points $x=0$ ($f=0$), $x=1$ ($f=-1$); endpoints $f(-1)=-5$, $f(2)=4$. Largest is 4.

Q41. Prove that $f(x) = 2x + 5$ is one-one by direct proof.

EASY

Answer: It is one-one.

Explanation: $f(x_1)=f(x_2)$ gives $2x_1+5=2x_2+5$, so $x_1=x_2$.

Q42. Disprove: 'every prime is odd'.

EASY

Answer: Counter-example 2.

Explanation: 2 is prime but even, so the statement is false.

Q43. Use contradiction to argue $\sqrt{2}$ is irrational (outline).

HARD

Answer: Assume rational, reach contradiction.

Explanation: Assume $\sqrt{2}=p/q$ in lowest terms; then $p^2=2q^2$ forces p and q both even, contradicting lowest terms.

Q44. State the contrapositive of 'if n^2 is even then n is even'.

MEDIUM

Answer: If n is odd then n^2 is odd.

Explanation: Contrapositive of $(p \Rightarrow q)$ is $(\text{not } q \Rightarrow \text{not } p)$.

Q45. Model the height of a tower from elevation angle α , distance l , eye height h .

MEDIUM

Answer: $H = h + l \tan(\alpha)$.

Explanation: From the right triangle $\tan(\alpha) = (H-h)/l$, solved for H .

Q46. List the five steps of mathematical modelling.

EASY

Answer: Identify, translate, solve, interpret, refine.

Explanation: Identify the situation, convert to mathematics, solve, interpret the result, and refine assumptions if it disagrees with reality.

Q47. If $f(x) = x^3$ is strictly increasing, justify using its derivative.

MEDIUM

Answer: $f'(x) = 3x^2 \geq 0$.

Explanation: The derivative is non-negative everywhere (and zero only at one point), so f is strictly increasing on \mathbb{R} .

Q48. Find dy/dx for $y = (\sin x)^x$.

HARD

Answer: $(\sin x)^x (\log \sin x + x \cot x)$.

Explanation: $\log y = x \log \sin x$; differentiate: $(1/y)y' = \log \sin x + x \cot x$.

Q49. Evaluate $\sin(\tan^{-1} x)$ for $|x| < 1$.

MEDIUM

Answer: $x/\sqrt{1+x^2}$.

Explanation: Let $\tan \theta = x$; opposite x , adjacent 1, hypotenuse $\sqrt{1+x^2}$; $\sin \theta = x/\sqrt{1+x^2}$.

Q50. For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, verify $|A|$ and state if A is invertible.

EASY

Answer: $|A| = -2$, invertible.

Explanation: $ad - bc = 4 - 6 = -2$, which is non-zero, so A is non-singular and invertible.